A Solution to the $3x + 1$ Problem

by

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“Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done.”¹

“One of the greatest contributions a mathematician can make is to spot something so simple and powerful that everybody else has missed it.”²

Note 1: We have discovered that the proof that was in Appendix F contained a fatal error, so we have deleted the Appendix. Our other two proofs of the $3x + 1$ Conjecture remain correct as far as we know. These proofs are given under: “Theorem: The $3x + 1$ Conjecture is true.” on page 19 and in “Appendix H — Second Proof of the $3x + 1$ Conjecture” on page 44. The first 11 pages of this paper provide sufficient background to understand either of these proofs.

Note 2: The reader can safely assume, initially, that all referenced lemmas in this paper are true, since their proofs have been checked and deemed correct by several mathematicians.

Note 3: We will offer shared-authorship to any mathematician who creates a proof of the $3x + 1$ Conjecture that differs from those in this paper, but that makes use of materials in this paper.

Note 4: The author is seeking a professional mathematician to help prepare this paper for publication. The author will pay any reasonable consulting fee, give generous credit in the Acknowledgments (but only with the mathematician’s prior written approval), and offer shared-authorship for significant contribution to content.

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Abstract

The 3x + 1 Problem asks if repeated iterations of the function \( C(x) = (3x + 1)/(2^a) \) always terminate in 1. Here \( x \) is an odd, positive integer, and \( a = \text{ord}_2(3x + 1) \), the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the 3x + 1 Conjecture. An odd, positive integer that maps to 1 is called a non-counterexample; an odd, positive integer that doesn’t map to 1 is called a counterexample (to the Conjecture).

Our first proof is based on a structure called tuple-sets that represents the 3x + 1 function in the “forward” direction. In our proof, we show that the 35-level elements of all 35-level tuples in all 35-level tuple-sets are the same, regardless if counterexamples to the Conjecture exist or not\(^1\). From this fact, a simple inductive argument allows us to conclude that all tuple-sets are the same, whether counterexamples exist or not, and hence that counterexamples do not exist.

Our second proof is based on a structure called the 1-tree, which is at least part of the 3x + 1 function in the “inverse” direction. It represents all odd, positive integers that map to 1. We show that, counter-intuitively, there is only one 1-tree, and from that fact we derive our proof.

Our third proof, like the first, is based on tuple-sets. In this proof, we define anchor, which is the \( i \)-level element of the first \( i \)-level tuple in an \( i \)-level tuple-set. We then show that there is one and only one set of anchors for all \( i \), regardless if counterexamples exist or not. We then show that this implies that there is one and only one set of infinite tuples, regardless if counterexamples exist or not, and from this we deduce that, if counterexamples exist, then some infinite tuples must be both counterexample and non-counterexample tuples, which is absurd, hence counterexamples do not exist and the Conjecture is true.

As far as we have been able to determine, our approaches to a solution of the Problem are original.

\(^1\) A phrase of the form “\( q \) regardless if \( p \)” is equivalent to “(if \( p \) then \( q \)) and (if not-\( p \) then \( q \))”. It is meaningful and in fact true as long as \( q \) is true, which it always is in this paper. Instances of the phrase occur in everyday speech, for example, “Fermat’s Last Theorem is true regardless if the Riemann Conjecture is true”,.
Introduction

Statement of Problem

For $x$ an odd, positive integer, set

$$C(x) = \frac{3x + 1}{2^{\text{ord}_2(3x + 1)}}$$

where $\text{ord}_2(3x + 1)$ is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example, $C(17) = 13$ ($\text{ord}_2(3(17) + 1) = 2$), $C(13) = 5$ ($\text{ord}_2(3(13) + 1) = 3$), $C(5) = 1$ ($\text{ord}_2(3(5) + 1) = 4$). Each of these constitutes one iteration of the $3x + 1$ function. The $3x + 1$ Problem, also known as the $3n + 1$ Problem, the Syracuse Problem, Ulam’s Problem, the Collatz Conjecture, Kakutani’s Problem, and Hasse’s Algorithm, asks if repeated iterations of $C$ always terminate at 1. The conjecture that they do is hereafter called the $3x + 1$ Conjecture, or sometimes, in this paper, just the Conjecture. We call $C$ the $3x + 1$ function; note that $C(x)$ is by definition odd.

An odd, positive integer such that repeated iterations of $C$ terminate at 1, we call a non-counterexample. An odd, positive integer such that repeated iterations of $C$ never terminate at 1, we call a counterexample.

Other equivalent formulations of the $3x + 1$ Problem are given in the literature; we base our formulation on the $C$ function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

Summary of Research on the Problem

As stated in (Lagarias 1985), “The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930’s, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions. In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals...”

Lagarias has performed an invaluable service to the $3x + 1$ research community by publishing several annotated bibliographies relating to the Problem. These are accessible on the Internet.

On the Structure of This Paper

To enhance readability, we have placed proofs of all lemmas in “Appendix A — Statement and Proof of Each Lemma” on page 22.
In Memoriam

Several of the most important lemmas in this paper (though not Lemma 1.0) were originally conjectured by the author and then proved by the late Michael O’Neill. He made a major contribution to this research, and is sorely missed.
Tuple-sets: The Structure of the 3x + 1 Function in the “Forward” Direction

A Comment

The structure, tuple-sets, that we are about to describe, is one of two remarkably simple structures underlying the 3x + 1 function, a function that is still referred to, at least informally, as “chaotic”. The reader can get an idea of the alternative structures that, at the time of this writing, are used throughout the 3x + 1 literature, by browsing papers in Google that come up in response to the search string, “Collatz graphs”. For example, see:
https://www.fq.math.ca/Scanned/40-1/andaloro.pdf,
http://go.helms-net.de/math/collatz/aboutloop/collatzgraphs.htm

Brief Description of Tuple-sets

The following should be sufficient for the reader to understand our proofs of the 3x + 1 Conjecture that are based on tuple-sets, namely those in “Theorem: The 3x + 1 Conjecture is true.” on page 19 and “Appendix H — Second Proof of the 3x + 1 Conjecture” on page 44.

1. We use the definition of the 3x + 1 function in which all successive divisions by 2 are collapsed into a single exponent of 2 (see “Statement of Problem” on page 3). Thus, for example, the tuple <9, 7, 11> represents the fact that
9 maps to 7 in one iteration of the function, via the exponent 2, because \((3(9) + 1)/2^2 = 7\);
7 maps to 11 in one iteration of the function, via the exponent 1, because \((3(7) + 1)/2^1 = 11\).

2. We see that the sequence of exponents associated with the tuple <9, 7, 11> is \{2, 1\}.

3. A tuple-set \(T_A\) is the set of all finite tuples that are associated with the exponent sequence \(A\) (and “approximations” to \(A\), but this is not important for our proofs of the 3x + 1 Conjecture). In our example, \(A = \{2, 1\}\).

   In addition to the tuple <9, 7, 11>, the tuple-set \(T_A = T_{\{2, 1\}}\) contains the tuples <25, 19, 29>, <41, 31, 47>, and an infinity of others, each associated with the exponent sequence \{2, 1\}. (See “Lemma 1.0: the “Distance” Functions \(d(i, i)\) and \(d(1, i)\)” on page 9.)

4. Facts about tuple-sets:

   An \(i\)-level tuple-set \(T_A, i \geq 2\), contains (among other tuples, see previous step) all \((i + 1)\)-element tuples that are associated with the exponent sequence \(A\).

   There is an infinity of tuples in each tuple-set.

   The set of all tuple-sets contains tuples representing all finite iterations of the 3x + 1 function.
A Solution to the 3x + 1 Problem

Full Description of Tuple-sets

Definitions

Iteration

An iteration takes an odd, positive integer, \( x \), to an odd, positive integer, \( y \), via one application of the \( 3x + 1 \) function, \( C \). Thus, in one iteration \( C \) takes 17 to 13 because \( C(17) = 13 \).

Tuple

A (finite) tuple is a finite sequence of zero or more successive iterations of \( C \), that is, \(< x, C(x), C^2(x), ..., C^k(x)>\), where \( k \geq 0 \).

A finite tuple is the prefix of an infinite tuple. If \( x \) is a non-counterexample, then \( x \) is the first element of an infinite tuple \(< x, y, ..., 1, 1, 1, ... >\). Of course, if \( x \) is a range element of \( C \), then \( x \) can be an element other than the first in another non-counterexample tuple.

In the literature, a tuple (finite or infinite) is usually called a trajectory or an orbit.

If \( x \) is a counterexample, then \( x \) is the first element of an infinite tuple \(< x, y, ... >\) which does not contain 1. Of course, if \( x \) is a range element of \( C \), then \( x \) can be an element other than the first in another counterexample tuple.

A counterexample tuple must be one of two types: either there is an infinitely-repeated finite cycle of elements (none of which is 1) in the infinite tuple having the counterexample \( x \) as first element, or else there is no such cycle, but there is no 1 in the infinite tuple having the counterexample \( x \) as first element — in other words, there is no upper bound to the elements of the infinite tuple.

Exponent, Exponent Sequence

If \( C(x) = y \), with \( y = (3x + 1)/2^a \), we say that \( a \) is the exponent associated with \( x \). In more formal language, this can be expressed as \( \text{ord}_2(3x + 1) = a \). Sometimes we simply write \( e(x) = a \).

The sequence \( \{a_2, a_3, ..., a_i\} \), where \( a_2, a_3, ..., a_i \) are the exponents associated with \( x, C(x), ..., C^{(i-1)}(x) \) respectively, is called an exponent sequence. We number exponents beginning with \( a_2 \) in order that the subscript corresponds to a level number in the corresponding tuple-set. See “Levels in Tuples and Tuple-sets” on page 7

We say that \( x \) maps to \( y \) via \( a_i \) if \( C(x) = y \) and \( \text{ord}_2(3x + 1) = a_i \). By extension, we say that \( x \) maps to \( z \) if \( z \) is the result of a finite sequence of iterations of \( C \) beginning with \( x \), that is if the tuple \(< x, y, ..., z >\) exists.
Tuple-set

Let \( A = \{a_2, a_3, \ldots, a_i\} \) be a finite sequence of exponents, where \( i \geq 2 \). The tuple-set \( T_A \) consists of all and only the tuples that are associated with all successive approximations to \( A \). Thus \( T_A \) consists of all and only the following tuples. (Note: First elements \( x \) in different tuples are different odd, positive integers. No two tuples in a tuple-set have the same first element.)

- all tuples \( <x> \) such that \( x \) does not map to an odd, positive integer via \( a_2 \);
- all tuples \( <x, y> \) such that \( x \) maps to \( y \) via \( a_2 \) but \( y \) does not map to an odd, positive integer via \( a_3 \);
- all tuples \( <x, y, y'> \) such that \( x \) maps to \( y \) via \( a_2 \) and \( y \) maps to \( y' \) via \( a_3 \), but \( y' \) does not map to an odd, positive integer via \( a_4 \);
- ...
- all tuples \( <x, y, y', \ldots, y^{(i-3)}, y^{(i-2)}> \) such that \( x \) maps to \( y \) via \( a_2 \) and \( y \) maps to \( y' \) via \( a_3 \) and ... and \( y^{(i-3)} \) maps to \( y^{(i-2)} \) via the exponent \( a_i \). (The longest tuple in an \( i \)-level tuple-set has \( i \) elements.)

In other words, for each \( i \)-level exponent sequence \( A \):

- there are tuples \( <x> \) whose associated exponent sequence is a prefix of \( A \) for no exponent of \( A \), and
- there are other tuples \( <x, y> \) whose associated exponent sequence is a prefix of \( A \) for the first exponent of \( A \), and
- there are other tuples \( <x, y, y'> \) whose associated exponent sequence is a prefix of \( A \) for the first two exponents of \( A \), and
- ...
- there are other tuples \( <x, y, z, \ldots, y^{(i-2)}> \) whose associated exponent sequence is a prefix of \( A \) for all \( i - 1 \) exponents of \( A \).

Tuples are ordered in the natural way by their first elements.

The set of first elements of all tuples in a tuple-set is the set of odd, positive integers (see proof under “The Structure of Tuple-sets” on page 9). Thus, there is a countable infinity of tuples in each tuple-set.

For each \( i \geq 2 \), tuple-sets are a partition of the set of all \( i \)-level tuples.

Levels in Tuples and Tuple-sets

Let \( A \) be an \( i \)-level exponent sequence, \( \{a_2, a_3, \ldots, a_i\} \). The reason subscripts of exponents begin with 2, rather than with 0 or 1, is so that they correspond to levels in each tuple-set. (No
tuple-set has only one level, because that would mean it is associated with no exponent sequence.) Let \( T_A \) be the tuple-set determined by \( A \). Then, by definition of tuple-set, there exist \( j \)-level tuples in \( T_A \), where \( 1 \leq j \leq i \), that is, tuples \( t = \langle x, y, ..., z \rangle \), where \( x \) is the 1-level element of \( t \), \( y \) is the 2-level element of \( t \), ..., and \( z \) is the \( j \)-level element of \( t \). We say that \( T_A \) is an \( i \)-level tuple-set, and we sometimes speak of the set of \( j \)-level tuple-elements in \( T_A \), where \( 1 \leq j \leq i \).

For \( 2 \leq j \leq i \), two tuples are said to be \textit{consecutive at level} \( j \) if no \( j \)-level or higher-level tuple exists between them.

**Example of Tuple-set**

As an example of (part of) a tuple-set: in Fig. 1, where \( A = \{a_2, a_3, a_4\} = \{1, 1, 2\} \) and where we adopt the convention of orienting tuples vertically on the page, the tuple-set \( T_A \) includes:

- the tuple \( \langle 1 \rangle \), because \( e(1) = 2 \neq (a_2 = 1) \);
- the tuple \( \langle 3, 5 \rangle \), because \( e(3) = (a_2 = 1) \), but \( e(5) = 4 \neq (a_3 = 1) \);
- the tuple \( \langle 5 \rangle \), because \( e(5) = 4 \neq (a_2 = 1) \);
- the tuple \( \langle 7, 11, 17, 13 \rangle \) because \( e(7) = 1 \), \( (a_2 = 1) \) and \( e(11) = 1 \), \( (a_3 = 1) \) and \( e(17) = 2 \) \( (a_4 = 2) \);
- etc.

\[
\begin{array}{ccccccccccc}
\text{level} & 4 & 3 & 2 & 1 \\
\hline
4 & 13 & 17 & 5 & 1 \\
3 & 18 & 35 & 11 & 7 \\
2 & 23 & 29 & 17 & 9 \\
1 & 19 & 21 & 13 & 1 \\
\end{array}
\]

**Fig. 1.** Part of the tuple-set \( T_A \) associated with the sequence \( A = \{1, 1, 2\} \)

The number 18 between the arrows at level 3 and the number 4 between the arrows at level 1 are the values of the level 3 and level 1 distance functions, respectively, established by Lemma 1.0 (see “Lemma 1.0: the “Distance” Functions \( d(i, i) \) and \( d(1, i) \)” on page 9).

In each \( i \)-level tuple-set \( T_A \), where \( i \geq 2 \), for each odd, positive integer \( x \) there exists a tuple whose first element is \( x \). The tuple may be one-level \( \langle x \rangle \), or 2-level \( \langle x, y \rangle \), or ... or \( i \)-level \( \langle x, y, y', ..., y^{(i-3)}, y^{(i-2)} \rangle \). Thus each tuple-set is non-empty.
Graphical Representation of the Set of All Tuple-sets

It is clear from the definition of tuple-set that the set of all tuple-sets can be represented by an infinitary tree in which each node is a tuple-set. We can imagine the tuple-set (which contains an infinity of tuples) extending into the page.

The Structure of Tuple-sets

It is important for the reader to understand that the structure of each tuple-set is unchanged by the presence or absence of counterexample tuples. Regardless if counterexample tuples exist or not, the set of first elements of all tuples in each tuple-set is always the same, namely, the set of odd, positive integers. Proof: Let \( x \) be any odd, positive integer and let \( A = \{a_2, a_3, ..., a_i\} \), where \( i \geq 2 \), be any exponent sequence. Then there are exactly two possibilities:

1. \( x \) maps to a \( y \) in a single iteration of the \( 3x + 1 \) function, \( C \), via the exponent \( a_2 \), or
2. \( x \) does not map to a \( y \) in a single iteration of \( C \) via the exponent \( a_2 \).

But if (1) is true, then a tuple containing at least two elements, with \( x \) as the first, is in \( T_A \); if (2) is true, then the tuple \( <x> \) is in \( T_A \). There is no third possibility. □

For each tuple-set, the first element of the first tuple is 1, the first element of the second tuple is 3, the first element of the third tuple is 5, etc.

It can never be the case that, if counterexample tuples exist, then somehow there are “more” tuples in a tuple-set than if there are no counterexample tuples\(^1\).

Furthermore, the distance functions defined in “Lemma 1.0: the “Distance” Functions \( d(i, i) \) and \( d(1, i) \)” on page 9 are the same regardless if counterexample tuples exist or not.

Extensions of Tuple-sets

Since there is a tuple-set for each finite sequence \( A \) of exponents, it follows that each tuple-set \( T_A \) has an extension via the exponent 1, and an extension via the exponent 2, and an extension via the exponent 3, ... In other words, if \( A = \{a_2, a_3, ..., a_i\} \), then there is a tuple-set \( T_A' \), where \( A' = \{a_2, a_3, ..., a_i, 1\} \), and a tuple-set \( T_A'' \), where \( A'' = \{a_2, a_3, ..., a_i, 2\} \), and a tuple-set \( T_A''' \), where \( A''' = \{a_2, a_3, ..., a_i, 3\} \), ...

All this is true whether or not the tuple-set \( T_A \) and/or any of its extensions contains counterexample tuples or not.

For further details on extensions of tuple-sets, see “How Tuple-sets ‘Work’” and the proof that there exists an extension for each tuple-set (“Lemma 3.0 Statement and Proof”) in our paper, “Are We Near a Solution to the \( 3x + 1 \) Problem?” on occampress.com.

Lemma 1.0: the “Distance” Functions \( d(i, i) \) and \( d(1, i) \)

(a) Let \( A = \{a_2, a_3, ..., a_i\} \), where \( i \geq 2 \), be a sequence of exponents, and let \( t_{(r)}, t_{(s)} \) be tuples consecutive at level\(^2\) \( i \) in \( T_A \). Then \( d(i, i) \) is given by:

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\(^1\) To make this statement more precise: in no tuple-set does there ever exist a first element of a tuple, regardless how large that first element is, such that there are more tuples in that tuple-set having smaller first elements if counterexamples exist, than if counterexamples do not exist.
A Solution to the $3x + 1$ Problem

\[ d(i, i) = 2 \cdot 3^{(i-1)} \]

(b) Let $t_{(r)}, t_{(s)}$ be tuples consecutive at level $i$ in $T_4$. Then $d(1, i)$ is given by:

\[ d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \ldots (2^{a_i}) \]

**Proof:** see “Lemma 1.0: Statement and Proof” on page 22

It follows from part (a) of the Lemma that the set of all $i$-level elements of all $i$-level first tuples in all $i$-level tuple-sets is \( \{z \mid 1 \leq z < 2 \cdot 3^{i-1}\} \), where $z$ is an odd, positive integer not divisible by 3.

**Remark:** Relationships similar to those described in parts (a) and (b) of the Lemma hold for successive $j$-level tuples, where $2 \leq j < i$. The following table shows these relationships for $(i - j)$-level elements of tuples consecutive at level $(i - j)$ in an $i$-level tuple-set, where $0 \leq j \leq (i - 1)$. The distances are easily proved using Lemma 1.0.

### Table 1: Distances between elements of tuples consecutive at level $i$

<table>
<thead>
<tr>
<th>Level</th>
<th>Distance between $(i - j)$-level elements of tuples consecutive at level $(i - j)$, where $0 \leq j \leq (i - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$2 \cdot 3^{i-1}$</td>
</tr>
<tr>
<td>$i - 1$</td>
<td>$2 \cdot 3^{i-2} \cdot 2^{a_i}$</td>
</tr>
<tr>
<td>$i - 2$</td>
<td>$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} \cdot 2^{a_i}$</td>
</tr>
<tr>
<td>$i - 3$</td>
<td>$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} \cdot 2^{a_{i-1}} \cdot 2^{a_i}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>$2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}} \cdot 2^{a_i}$</td>
</tr>
<tr>
<td>1</td>
<td>$2 \cdot 2^{a_2} \cdot 2^{a_3} \ldots 2^{a_{i-1}} \cdot 2^{a_i}$</td>
</tr>
</tbody>
</table>

Further details can be found in the section, “Remarks About the Distance Functions” in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com.

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2. For $2 \leq j \leq i$, two tuples are consecutive at level $j$ if no $j$-level or higher-level tuple exists between them (see “Levels in Tuples and Tuple-sets” on page 7.)
Lemma 2.0

Assume a counterexample exists. Then for all $i \geq 2$, each $i$-level tuple-set contains an infinity of $i$-level counterexample tuples and an infinity of $i$-level non-counterexample tuples.

Proof: “Lemma 2.0: Statement and Proof” on page 27
Recursive “Spiral”s: The Structure of the 3x + 1 Function in the “Backward”, or Inverse, Direction

The 1-Tree

Definition of the 1-Tree

We define a tree called the 1-tree as follows (see Fig. 4 below):

The set of all odd, positive integers that map to 1 in a single iteration\(^1\) of the 3x + 1 function is Level 1 in the tree; This set is \{1, 5, 21, 85, 341, \ldots\}. (Here and in all subsequent steps, we do not count iterations that take 1 to 1.)

The set of all odd, positive integers that map to 1 in two iterations of the 3x + 1 function is Level 2 in the tree;

The set of all odd, positive integers that map to 1 in three iterations of the 3x + 1 function is Level 3 in the tree;

... 

Thus the 1-tree contains all and only the odd, positive integers that map to 1 in a finite number of iterations of the function.

Definition of “Spiral”

We call the set of odd, positive integers that map to a range element \(y\) in one iteration of the function, a recursive “spiral”, or just a “spiral” for short. (See “Fig. 4. Recursive “spirals” structure of odd, positive integers that map to 1.” on page 13.) Thus, for example, \{1, 5, 21, 85, 341, \ldots\} is a “spiral”. It maps to 1 in one iteration of the 3x + 1 function. It is easily shown that the elements of a “spiral” map to \(y\) either by all odd exponents, or by all even exponents (which is the case with \{1, 5, 21, 85, \ldots\}). It is likewise easily shown that if \(x\) is an element of a “spiral”, then the next “spiral” element is 4\(x + 1\).

Each node in the 1-tree is a “spiral” element. From now on, we will use the term “spiral” element.

Each “spiral” element \(y\) that is a range element of the 3x + 1 function is the root of a \(y\)-tree. The definition of the \(y\)-tree is the same as that of the 1-tree, with \(y\) replacing 1.

The odd, positive integers between successive “spiral” elements we call an interval in the “spiral”. Thus, for example, in the “spiral” \{1, 5, 21, 85, 341, \ldots\}, 3 is the only element in the first interval; 7, 9, 11, 13, 15, 17, 19, are the elements of the second interval, etc.

If a counterexample exists, then it is an element of an interval, but it is never a “spiral” element. The tree or trees generated by counterexamples are of course separate (disjoint) from the 1-tree. There are “spiral”s, “spiral” elements, and intervals in counterexample trees just as there are in the 1-tree.

Let \(y\) denote a “spiral” element in \{1, 5, 21, 85, \ldots\} that is a range element. Then if \(x\) is a non-counterexample, it must be an element of a \(y\)-tree.

Suppose we ask, “What is the location (the “address”) of an element \(x\) in the 1-tree?” A simple answer is: the tuple <\(x\), \ldots, 1>.

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1. We remind the reader that throughout this paper we use, not the original definition of the 3x + 1 function, but Crandall’s equivalent definition, in which all successive divisions by 2 are collapsed into a single exponent of 2. See “Statement of Problem” on page 3.
So, to summarize: In the case of the 1-tree, all “spiral” elements are non-counterexamples. Some interval elements we know are non-counterexamples. If a counterexample exists, then it is an element of an interval, but never of a “spiral” in the 1-tree. In the case of a counterexample tree, all spiral elements are counterexamples. Some interval elements are counterexamples, others are non-counterexamples.

**Fig. 4. Recursive “spirals” structure of odd, positive integers that map to 1.**

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial “spirals” surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a $2^6$ because $(3 \cdot 1813 + 1)/2^6 = 85$. The line connecting 453 to 1813 is marked $85 \cdot 2^4$ because $453 + 85 \cdot 2^4 = 1813$. The quantity $85 \cdot 2^4 = 3 \cdot 453 + 1$, and similarly for the difference between successive elements of a “spiral” in all “spiral”s. These facts follow from the fact that if $x, y$ are consecutive elements of a “spiral”, with $x < y$, then $y = 4x + 1$.

*Not all elements of a “spiral” are mapped to by even exponents.* For example, 5 is mapped to by all odd exponents, as is 341. It is easily shown that successive elements of a “spiral” have the pattern ... $e, o, 3, e, o, 3, e, ...$, meaning that there is an element mapped by all *even* exponents, then an element mapped to by all *odd* exponents, then a multiple-of-3, which is not mapped to by any odd, positive integer, then an element mapped by all *even* exponents, then an element mapped to by all *odd* exponents, then a multiple-of-3, which is not mapped to by any odd, positive integer, then an element mapped by all *even* exponents, then 

... (The first element of a “spiral” can have the $e, o,$ or 3 property, but thereafter, the above pattern repeats for the entire “spiral”.)
The tree is not strictly an infinitary tree, because if a node is not a range element of the \(3x + 1\) function, that is, if a node is a multiple-of-3, then it has no descending nodes (no odd, positive integers map to a multiple-of-3).

There is exactly one 1-tree, whether or not counterexamples exist (“Lemma 3.0: Statement and Proof” on page 27). This means that an odd, positive integer maps to 1 regardless if counterexamples exist or not. Thus, for example, 13 (an element of the 5-tree) maps to 1 (in two iterations of the function) today. If a proposed proof of the \(3x + 1\) Conjecture is accepted as valid tomorrow, 13 will still map to 1. And if the Conjecture is proved false tomorrow, 13 will still map to 1. (A proof that there is only one 1-tree is the proof of “Lemma 3.0: Statement and Proof” on page 27.)

The reader may find the difference between the following two statements helpful:

(1) If \(x\) is a non-counterexample, then \(x\) is a non-counterexample whether or not counterexamples exist.

But

(2) If \(x\) is a counterexample, then \(x\) is most certainly not a counterexample whether or not counterexamples exist!

If the \(3x + 1\) Conjecture is true, then the 1-tree contains all odd, positive integers. If the Conjecture is false, then the 1-tree contains only a proper subset of the odd, positive integers.

Properties of the 1-Tree
(\textit{Note}: Some of the following properties are repeated from the above section.)

- There is exactly one 1-tree, whether or not counterexamples exist ( “Lemma 3.0: Statement and Proof” on page 27). The question is, Does the 1-tree contain all odd, positive integers (in which case the Conjecture is true) or does it contain only a proper subset of the odd, positive integers (in which case the Conjecture is false).

(The reader will find, in the Remarks in the section, “Lemma 3.0: Statement and Proof” on page 27, the elementary explanation of how the one and only 1-tree can be capable of containing all odd, positive integers or only a proper subset of the odd, positive integers.)

- The reader may find the difference between the following two statements helpful:

(1) If \(x\) is a non-counterexample, then \(x\) is a non-counterexample whether or not counterexamples exist.

But

(2) If \(x\) is a counterexample, then \(x\) is most certainly not a counterexample whether or not counterexamples exist!

- If \(x\) is an element of a “spiral”, then \(4x + 1\) is the next element of the “spiral”.

Each “spiral” maps to a range element \(y\) in one iteration of the \(3x + 1\) function either by all odd exponents, or by all even exponents.

The successive elements of a “spiral” are mapped to in accordance with a rule that can be expressed as ... 2, 1, 3, 2, 1, 3, ..., where “2” means “is mapped to by all even exponents”, “1” means “is mapped to by all odd exponents”, and “3” means “is not mapped to because element is
A Solution to the $3x + 1$ Problem

a multiple-of-x, hence not a range element”.

Thus, for example, in the “spiral” $S = \{1, 5, 21, 85, ... \}$, 1 is mapped to by all even exponents, 5 is mapped to by all odd exponents, 21 is not mapped to because it is a multiple-of-3, 85 is mapped to by all even exponents, ...

- The elements of each “spiral” other than $S$, occupy an infinity of successive intervals in $S$, one element per interval. Thus, consider the “spiral” $\{3, 13, 53, 213, ... \}$. The same applies recursively down through the 1-tree, except that if a “spiral” $s$ maps via even exponents to $y$ in another “spiral $r$” in one iteration of the $3x + 1$ function, then the elements of $s$ skip one interval in $r$ before occupying an infinity of successive intervals in $r$, one element per interval.

- Each “spiral” element $x$, and the range element $y$ it maps to in one iteration of the $3x + 1$ function, is a tuple $<x, y>$ in a tuple-set. Since the set of elements of a “spiral” map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all “spiral”s in the 1-tree is the set of all first elements of all non-counterexample 2-tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

- The following table shows the first three 2-tuples in the 2-level tuple-sets $T_A$ for $A = \{1\}$, $\{3\}$, and $\{5\}$. (The reader might find it more natural to imagine the exponent 1 tuple-set as being foremost, with the exponent 3 tuple-set parallel and directly behind the exponent 1 tuple-set, and the exponent 5 tuple-set as parallel and directly behind the exponent 3 tuple-set, etc.)

In brief, “spiral” elements run vertically upward, tuple-set elements run horizontally to the right.

Observe the first three elements of the “spiral” that maps to 5 in one iteration of the $3x + 1$ function, namely, the elements 3, 13, 53 running vertically (and upward in the Table).

Observe the first three elements of the “spiral” that maps to 11 in one iteration of the $3x + 1$ function, namely, the elements 7, 29, 117 running vertically (and upward).

Observe the first three elements of the “spiral” that maps to 17 in one iteration of the $3x + 1$ function, namely, the elements 11, 45, 181 running vertically (and upward).

In keeping with the rule governing successive elements of “spiral”s, namely, that if $x$ is a “spiral” element, then $4x + 1$ is the next element, we observe that $3(4) + 1 = 13$, etc.; $4(7) + 1 = 29$, etc., and $11(4) + 1 = 45$, etc.

But observe also, in keeping with “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 9 :

the (horizontal) sequence of level-2 elements in each 2-level tuple-set is 5, 6, 11, ...

The (horizontal) difference between level-1 elements in the tuple-set $T_A$ where $A = \{1\}$ is $2(2^1)$,

The (horizontal) difference between level-1 elements in the tuple-set $T_A$ where $A = \{3\}$ is $2(2^3)$,

The (horizontal) difference between level-1 elements in the tuple-set $T_A$ where $A = \{5\}$ is $2(2^5)$,
A Solution to the 3x + 1 Problem

Table 2: Partial View of Relationship Between 2-Level Tuple-sets and Odd-Exponent “Spiral”s

<table>
<thead>
<tr>
<th>Exponent</th>
<th>5</th>
<th>11</th>
<th>17</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>53</td>
<td>117</td>
<td>181</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>11</td>
<td>17</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>29</td>
<td>45</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>11</td>
<td>17</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7</td>
<td>11</td>
<td>...</td>
</tr>
</tbody>
</table>

Similar 2-level tuple-sets and corresponding “spiral” elements exist for even exponents, that is, for 2-level tuple-sets $T_A$ where $A = \{2\}, \{4\}, \{6\}, \ldots$.

- If counterexamples exist, it is not because a sub-tree of the 1-tree has somehow been “broken off” and become a counterexample tree. The structure of the 1-tree is the same, whether or not counterexamples exist.

- If counterexamples exist, then the intervals in each “spiral” — not just the “spiral” $\{1, 5, 21, 85, \ldots\}$ — contain an infinity of counterexamples (“Lemma 2.0” on page 11).

- It is not possible to tell, from a single “spiral”, if counterexamples exist or not. For each range element $y$ of the $3x + 1$ function, there is exactly one “spiral”, whether or not counterexamples exist. The function is not itself a function of the existence or non-existence of counterexamples.

- Is there a contradiction in the fact that, in the tree(s) of counterexamples, the intervals of each “spiral” contain all the non-counterexamples?

- It is possible to traverse from any point (“spiral” element) $y$ in the 1-tree to any other point (“spiral” element) $z$ in the 1-tree via repetitions of the following movements:
  - one element to the right or to the left of $y$ (the latter is possible only if $y$ is not the first element of the “spiral”);
  - up one element to the range-element $u$ that the “spiral” maps to in one iteration of the $3x + 1$ function, or...
down one element to the range element \( v \) that maps to \( y \) in one iteration of the function (there is no such \( v \) if \( y \) is a multiple-of-3, since then \( v \) is not a range element).

- Suppose we ask, “What is the location (the “address”) of an element \( x \) in the 1-tree?” A simple answer is: the tuple \(<x, \ldots, 1>\!

**Understanding the 1-Tree Relative to Successive Non-Counterexamples Beginning With 1**

1. It is known by computer test\(^1\) that all successive odd, positive integers less than approximately \( 5.76 \times 10^{18} \) are non-counterexamples. Let \( W \) denote the set of all odd, positive integers less than \( 10^{18} \). So let us consider the “spiral” \( S = \{1, 5, 21, 85, \ldots \} \), all of whose elements map to 1 in one iteration of the \( 3x + 1 \) function, and proceed as follows.

2. **Definition:** If \( y \) is a range element (that is, not a multiple-of-3), then \( y \) is the root of a \( y \)-tree, which has the same structure and properties as the 1-tree, except, of course, that the root is \( y \), and not 1. Thus a \( y \)-tree is an infinitary, infinitely-deep tree.

3. We can color green all elements of all \( y \)-trees such that \( y \) is a range element that is an element of \( W \).

The green elements are all those odd, positive integers that map to 1 as a result of the computer tests having shown that at least all odd, positive integers less than \( 10^{18} \) map to 1. There is an infinity of green elements because they are elements of \( y \)-trees, and each \( y \)-tree is infinitary and infinitely deep.

One fact that might be promising for a proof of the \( 3x + 1 \) Conjecture is that each successive element of each ‘spiral’ is an element of an infinity of successive intervals in the “spiral” \( S = \{1, 5, 21, 85, 341, \ldots \} \). Thus, consider the elements of the “spiral” \( \{3, 13, 53, 213, \ldots \} \).

A proof that the elements of a “spiral” occupy an infinity of successive intervals in the “spiral” \( S = \{1, 5, 21, 85, 341, \ldots \} \) will be found in our paper, “Are We Near a Solution to the \( 3x + 1 \) Problem?”, on occampress.com, in the sub-section, “‘Spiral’s, Intervals and Levels” of the section, “Strategy of ‘Filling-in’ of Intervals in the Base Sequence Relative to 1”.

In the case of the 1-tree for the \( 3x – 1 \) function, the first counterexample, 5, occurs in the first interval in the “spiral” \( \{1, 3, 11, 43, \ldots \} \) that maps to 1 in one iteration of the \( 3x – 1 \) function — namely, in the interval whose elements are 5, 7, 9. (The element 7 is also a counterexample, because \(<5, 7, 5, \ldots>\) is an infinite cycle that never yields 1.)

---

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than \( 20 \times 2^{58} \approx 5.76 \times 10^{18} \), which is greater than \( 3.33 \times 10^{16} \approx 2 \times 3^{35 - 1} \), have been tested and found to be non-counterexamples.
**Theorem:** The $3x + 1$ Conjecture is true.

**Proof:**

(Note 1: Another proof of the Conjecture is given in “Appendix H — Second Proof of the $3x + 1$ Conjecture” on page 44.)

(Note 2: we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

We show that there is no difference (in form or content) between tuple-sets if counterexamples do not exist, and tuple-sets if counterexamples exist. From that fact, we conclude that counterexamples do not exist.

1. It is easily shown that, for each $i \geq 2$, the set $E_i$ of $i$-level elements in first $i$-level tuples in all $i$-level tuple-sets is the set of odd, positive integers less than $2 \cdot 3^{(i-1)}$ that are not divisible by 3 (by part (a) of “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 9). Thus, for example, $E_2$ is the set of all the odd, positive integers less than $2 \cdot 3^{(2-1)} = 6$, that are not divisible by 3, and these integers are 1 and 5.

2. By computer test, it is known that $E_2, E_3, E_4, \ldots$, up to at least $E_{35}$ each consists solely of non-counterexamples$^1$.

3. (1) For each 35-level tuple-set $T_A$, the sequence $S$ of 35-level elements in the sequence of 35-level tuples is given by $y + n(2 \cdot 3^{(35-1)})$, where $n \geq 0$ and $y$ is the 35-level element of the first 35-level tuple in the tuple-set $T_A$. The sequence $S$ is the sequence if counterexamples do not exist, and it is also the sequence if counterexamples exist. As we stated in step 2, $y$ is a non-counterexample element.

   **Proof:** Follows from part (a) of “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 9. The Distance Functions are not themselves functions of the truth or falsity of the $3x + 1$ Conjecture. □

   **Note:** the fact that all elements of $E_{35}$ are non-counterexamples is emphatically not the case for the $3x – 1$ function, where one of the elements, 5, of $E_2$ is already a counterexample. Thus there exists a first 2-level tuple, namely <7, 5>, in a 2-level tuple-set that is a counterexample tuple. Each subsequent $E_i$ contains counterexamples, each of which is the $i$-level element of the first $i$-level tuple in an $i$-level tuple-set. Each of these tuples is therefore a counterexample tuple. So our proof cannot be used to prove the false $3x – 1$ Conjecture$^2$.

4. Since each 35-level tuple-set— and indeed each $i$-level tuple-set, where $i \geq 2$ — has an

---

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)}$, have been tested and found to be non-counterexamples. These include the set of all 35-level elements of all 35-level first tuples in all 35-level tuple-sets.

2. See “Appendix I — The $3x – 1$- and Related Functions” on page 47.
extension via each possible exponent, namely, via 1, 2, 3, ..., we can use an inductive argument beginning with (1) to arrive at the conclusion that the set of all non-counterexample tuples if counterexamples do not exist, is the same as the set of all non-counterexample tuples if counterexamples exist.

5. We must now ask if counterexamples can exist in $T_A$ in $j$-level tuples, where $j < 35$. The answer is No, because each $j$-level tuple in $T_A$ extends to a 35-level tuple in some other 35-level tuple-set, and $T_A$ is any 35-level tuple-set. From step 3 we know that there is one and only one sequence $S$ of 35-level elements in the sequence of 35-level tuples in any tuple-set. Thus if any $j$-level tuples are counterexample tuples, they always behave the same as non-counterexample tuples.

6. So we must conclude from step 4 that counterexamples behave the same as non-counterexamples, which is absurd, or else that the set of counterexamples is the empty set. In either case, our conclusion must be that the $3x + 1$ Conjecture is true. □

**Remark 1**

The reader might enjoy answering — or attempting to answer — question (I), below, which arises from the following facts:

Let the tuple $t = <x, ..., 1, 1, ..., 1>$, which is clearly a non-counterexample tuple ($x$ is a non-counterexample). Let $A$ be the $i$-level exponent sequence associated with $t$. Then $t$ is the first $i$-level tuple in the tuple-set $T_A$. The 1-level (first) element of $t$ is $x$, the $i$-level element of $t$ is 1.

The 1-level (first) element of the $n$th $i$-level tuple in $T_A$ is given by

$$x + (n - 1)(2 \cdot (2^{a_2})(2^{a_3})...(2^{a_i})),$$

and the $i$-level element of the $n$th $i$-level tuple in $T_A$ is given by

$$1 + (n - 1)(2 \cdot 3^{i-1}),$$

where $n \geq 1$ (by parts (b) and (a) of “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 9).

(I)

How does $T_A$ differ if (1) counterexamples exist, and (2) counterexamples do not exist?

**Remark 2**

The reader might enjoy reading at least some of the appendices below that give other possible proofs of the $3x + 1$ Conjecture.

**Remark 3**

A wealth of additional results and possible strategies is available in our paper, “Are We Near a Solution to the $3x + 1$ Problem?” on occampress.com.
References


Lemma 1.0: Statement and Proof

Definition: Let $T_A$ be an $i$-level tuple-set, where $i \geq 2$. Let $t(r)$, $t(s)$ denote tuples consecutive at level $i$, with $r < s$ in the natural ordering of tuples by first elements. Let $t(r)(h)$, $t(s)(h)$ denote the elements of $t(r)$, $t(s)$ at level $h$, where $1 \leq h \leq i$. Then we call $|t(s)(h) - t(r)(h)|$ the distance between $t(r)$ and $t(s)$ at level $h$. We denote this distance by $d(h, i)$ and call $d$ the distance functions (one function for each $h$).

Lemma 1.0

(a) Let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, be a sequence of exponents, and let $t(r)$, $t(s)$ be tuples consecutive at level $i$ in $T_A$. Then $d(i, i)$ is given by:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let $t(r)$, $t(s)$ be tuples consecutive at level $i$ in $T_A$. Then $d(1, i)$ is given by:

$$d(1, i) = 2 \cdot 2^{a_{2}}(2^{a_{3}})...(2^{a_{i}})$$

Thus, in “Fig. 1. Part of the tuple-set $T_A$ associated with the sequence $A = \{1, 1, 2\}$” on page 8, the distance $d(3, 3)$ between $t_{8}(3) = 35$ and $t_{4}(3) = 17$ is $2 \cdot 3^{3-1} = 18$. The distance $d(1, 2)$ between $t_{12}(1) = 23$ and $t_{10}(1) = 19$ is $2 \cdot 2^{1} = 4$.

Proof:
The proof is by induction.

Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:

Let $t(r)$ and $t(s)$ be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level $i = 2$ in the 2-level tuple-set $T_A$, where $A = \{a_2\}$. (See Fig. 2 (1).)
Then we have:

\[\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)}\]  \hspace{1cm} (1.1)

and since, by definition of \(d(1, 2)\),

\[t_{(s)(1)} = t_{(r)(1)} + d(1, 2)\]

we have:

\[\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)}\]  \hspace{1cm} (1.2)

Therefore, since, by definition of \(d(i, i)\),

\[t_{(r)(2)} + d(2, 2) = t_{(s)(2)}\]
we can write, from (1.1) and (1.2):

\[
\frac{3t_{(r)(1)} + 1}{2^{a_z}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_z}}
\]

By elementary algebra, this yields:

\[
2^{a_z}d(2, 2) = 3 \cdot d(1, 2)
\]

Now \(d(2, 2)\) must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level \(i\), it must be the smallest such even number, whence it follows that \(d(2, 2)\) must = \(3 \cdot 2\), and necessarily

\[
d(1, 2) = 2 \cdot 2^{a_z}
\]

A similar argument establishes that \(d(2, 2)\) and \(d(1, 2)\) have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

**Proof of Induction Step for Parts (a) and (b) of Lemma 1.0**

Assume the Lemma is true for all levels \(j \leq i\) and that \(T_A\) is an \(i\)-level tuple-set, where \(A = \{a_2, a_3, ..., a_i\}\).

Let \(t_{(r)}\) and \(t_{(s)}\) be tuples consecutive at level \(i\), and let \(t_{(r)}\) and \(t_{(f)}\) be tuples consecutive at level \(i + 1\). (See Fig. 2 (2).)
Then we have:

$$\frac{3t_r(i) + 1}{2^{a_{i+1}}} = t_r(i+1)$$

and since, by definition of $d(i, i)$,

$$t_f(i) = t_r(i) + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_r(i) + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_f(i+1)$$
Thus, since

\[ t_{(r)(i+1)} + d(i + 1, i + 1) = t_{(f)(i+1)} \]

we can write:

\[
\frac{3t_{t_{(f)(i+1)}} + 1}{2^{a_{i+1}}} + d(i + 1, i + 1) = \frac{3(t_{t_{(f)(i+1)}} + g d(i, i)) + 1}{2^{a_{i+1}}}
\]

This yields, by elementary algebra:

\[ 2^{a_{i+1}}d(i + 1, i + 1) = 3 \cdot gd(i, i) \]

As in the proof of the Basis Step, \(d(i+1, i+1)\) must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level \(i+1\), it must be the smallest such even number. Thus \(d(i+1, i+1) = 3 \cdot d(i, i)\), and

\[ g \cdot d(i, i) = 2^{a_{i+1}}d(i, i) .\]

Hence

\[ g = 2^{a_{i+1}} .\]

Now \(g\) is the number of tuples consecutive at level \(i\) that must be “traversed” to get from \(t_{(r)}\) to \(t_{(f)}\). By inductive hypothesis, \(d(1, i)\) for each pair of these tuples is:

\[ d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^{a_i} \]

hence, since

\[ g = 2^{a_{i+1}} \]

we have

\[ d(1, i + 1) = d(1, i) \cdot 2^{a_{i+1}} .\]

A similar argument establishes that \(d(i+1, i+1)\) and \(d(1, i+1)\) have the above values for every pair of tuples consecutive at level \(i+1\).

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. \(\square\)
Lemma 2.0: Statement and Proof

Assume a counterexample exists. Then for all \( i \geq 2 \), each \( i \)-level tuple-set contains an infinity of \( i \)-level counterexample tuples and an infinity of \( i \)-level non-counterexample tuples.

Proof:

1. Assume counterexamples exist. Then:

   There is a countable infinity of non-counterexample range elements.
   
   *Proof:* Each non-counterexample maps to a range element, by definition of range element.

   Each range element is mapped to by an infinity of elements
   ( "Lemma 6.0: Statement and Proof" on page 30). A countable infinity of these
   are range elements (proof of “Lemma 7.0: Statement and Proof” on page 32).

   There is a countable infinity of counterexample range elements.
   
   *Proof:* same as for non-counterexample case.

2. For each finite exponent sequence \( A \), and for each range element \( y \), non-counterexample or counterexample, there is an \( x \) that maps to \( y \) via \( A \) possibly followed by a buffer exponent ("Lemma 7.0: Statement and Proof" on page 32). The presence of the buffer exponent does not change the fact that \( x \) is the first element of a tuple associated with the exponent \( A \).

Lemma 3.0: Statement and Proof

Exactly one set \( J \) of odd, positive integers maps to 1, regardless if counterexamples exist or not. 1

First Proof:

The set \( J = \)

\[
\{ \text{odd, positive integers } y \mid y \text{ maps to } 1 \text{ in one iteration of the } 3x + 1 \text{ function} \} \cup \\
\{ \text{odd, positive integers } y \mid y \text{ maps to } 1 \text{ in two iterations of the } 3x + 1 \text{ function} \} \cup \\
\{ \text{odd, positive integers } y \mid y \text{ maps to } 1 \text{ in three iterations of the } 3x + 1 \text{ function} \} \cup \\
\ldots
\]

If an odd, positive integer \( y \) maps to 1, then it maps to 1 regardless if counterexamples exist or not. For, certainly \( y \) will continue to map to 1 if counterexamples do not exist. If counterexamples exist, and \( y \) does not continue to map to 1, that implies that the proof that counterexamples

---

1. Some readers consider the phrase “regardless if counterexamples exist or not” as unnecessary. The reason we include it is that when it was not present, other readers assumed that the meaning of the Lemma was: exactly one set \( J \) of odd, positive integers maps to 1 (if counterexamples do not exist), and exactly one (different) set \( J \) of odd, positive integers maps to 1 if counterexamples exist. This meaning is incorrect. A second reason for including the phrase will become clear when the reader reads our proofs of the 3x + 1 Conjecture.
exist somehow changes the definition of the $3x + 1$ function or the laws of arithmetic. But such changes are absurd. Thus the Lemma is true. □

**Remark 1**
An informal proof of the Lemma is to assert that it is just a generalization of facts like”

13 maps to 1 today;
if the $3x + 1$ Conjecture is proved true tomorrow it will still map to 1;
if the $3x + 1$ Conjecture is proved false tomorrow it will still map to 1.

**Remark 2**
The 1-tree exists “first”. Once it exists, we ask, “Which is true? (1) The 1-tree contains all the odd, positive integers, or (2) The 1-tree only contains a proper subset of the odd, positive integers.”

If the answer is (1), then counterexamples do not exist; if the answer is (2), then counterexamples exist.

**Remark 3**
For a long time we had been bothered by the question, How can one and the same 1-tree structure contain (possibly) the set of odd, positive integers, or (possibly) a proper subset of the set of odd, positive integers. Naively, we asked, If the latter, wouldn't there be “empty nodes” that would have been filled with odd, positive integers in the case that the 1-tree contained all odd, positive integers?

We had forgotten a basic fact of set theory, namely, that there can be a one-one match between the elements of an infinite set, and the elements of an infinite proper subset of that set. An example is the infinite set $\{1, 2, 3, 4, \ldots \}$ and the infinite proper subset $\{10, 20, 30, 40, \ldots \}$. Hence the infinite set of nodes of the 1-tree could hold the (infinite) set of odd, positive integers, or an (infinite) proper subset thereof.

**Lemma 4.0: Statement and Proof**

*No multiple of 3 is a range element.*

**Proof :**
If

$$\frac{3x + 1}{2^a} = 3m$$

then $1 \equiv 0 \mod 3$, which is false. □

**Lemma 5.0: Statement and Proof**

*Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the $3x + 1$ function.*
Proof:

Since the domain of the $3x + 1$ function is the odd, positive integers, the only relevant generators are $3(2k + 1), k \geq 0$. We show that, for each odd, positive integer $y$ not a multiple of 3, there exists a $k$ and an $a$ such that

$$y = \frac{3(3(2k + 1)) + 1}{2^a},$$

where $a$ is necessarily the largest such $a$, since $y$ is assumed odd.

Rewriting (11.1), we have:

$$y2^{a - 1} - 5 = 9k.$$  \hfill (11.2)

Without loss of generality, we can let $y \equiv r \mod 18$, where $r$ is one of $1, 5, 7, 11, 13, \text{ or } 17$ (since $y$ is odd and not a multiple of 3, these values of $r$ cover all possibilities mod 18). Or, in other words, for some $q, r, y = 18q + r$. Then, from (11.2) we can write:

$$18(2^{a - 1})q + (2^{a - 1})r - 5 = 9k.$$  \hfill (11.3)

Since the first term on the left-hand side is a multiple of 9, $(2^{a - 1})r - 5$ must also be if the equation is to hold. We can thus construct the following table. (Certain larger $a$ also serve equally well, but those given suffice for purposes of this proof.)

Table 3: Values of $r, a$, for Proof of Lemma

<table>
<thead>
<tr>
<th>$r$</th>
<th>$a$</th>
<th>$(2^{a - 1})r - 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>171</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>99</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>63</td>
</tr>
</tbody>
</table>

Given $q$ and $r$ (hence $y$), we can use $r$ to look up $a$ in the table, and then solve (11.3) for integral $k$, thus producing the multiple of 3 that maps to $y$ in one iteration of the $3x + 1$ function.  \hfill ☐
Lemma 6.0: Statement and Proof

(a) Each range element \( y \) is mapped to, in one iteration of the 3x + 1 function, by every exponent of one parity only. Furthermore,

(b) For each of the two parities, there exists a range element that is mapped to by every exponent of that parity.

Proof of part (a):
Steps 1 and 2 are slightly edited versions of proofs by Jonathan Kilgallin and Alex Godofsky. Any errors are entirely ours. Step 3 is a slightly edited version of a proof by Michael Klipper. Any errors are entirely ours.

1. We first show that if \( y \) is mapped to by the exponent \( a \), then \( y \) is mapped to by every exponent greater than \( a \) that is of the same parity as \( a \).

Let \( y \) be a range element, and let \( x \) map to \( y \) via the exponent \( a \). Then

\[
y = \frac{3x + 1}{2^a}
\]

We wish to show that there exists an \( x' \) such that \( x' \) maps to \( y \) via the exponent \( 2a + 2 \). That is, we wish to show that there exists an \( x' \) such that

\[
y = \frac{3x' + 1}{2^{a+2}}
\]

Rewriting, this gives

\[
x' = \frac{2^{a+2}y - 1}{3}
\]

Substituting for \( y \) yields

\[
x' = \frac{2^{a+2} \left( \frac{3x + 1}{2^a} \right) - 1}{3}
\]

Simplifying, this gives \( x' = 4x + 1 \). Since \( x \) is an odd, positive integer, clearly \( x' \) is as well.
Thus, by induction, if \( y \) is mapped to via the exponent \( a \), it is mapped to by every exponent greater than \( a \) of the same parity. \( \square \)

2. Next we show that if \( y \) is mapped to by the exponent \( a \) which is greater than 2, then it is mapped to by every exponent less than \( a \) that is of the same parity as \( a \).

Let \( y \) be a range element, and let \( x \) map to \( y \) via the exponent \( a \) where \( a > 2 \). Then

\[
y = \frac{3x + 1}{2^a}
\]

We wish to show that there exists an \( x' \) such that \( x' \) maps to \( y \) via the exponent \( 2^{a-2} \). That is, we wish to show that there exists an \( x' \) such that

\[
y = \frac{3x' + 1}{2^{a-2}}
\]

Rewriting, this gives

\[
x' = \frac{2^{a-2}y - 1}{3}
\]

Substituting for \( y \) yields

\[
x' = \frac{2^{a-2}\left(\frac{3x + 1}{2^a}\right) - 1}{3}
\]

Simplifying yields

\[
x' = \frac{x - 1}{4}
\]

3. We must now show that \( x' = (x - 1)/4 \) is an odd, positive integer. This means we must show that \( (x - 1) = 4(2k + 1) \) for some \( k \geq 0 \), or that \( (x - 1) = 8k + 4 \), hence that \( x = 8k + 5 \). Thus, we must prove \( x \equiv 5 \mod 8 \).
We know that $x$ maps to $y$ via $a$, where $a \geq 3$. Thus, $y = (3x + 1)/2^a$, so $2^ay = 3x + 1$. Because $a \geq 3$, $2^ay$ is a multiple of 8. Thus, $(3x + 1) \equiv 0 \mod 8$, and $3x \equiv 7 \mod 8$. This readily implies $x \equiv 5 \mod 8$.

4. Thus, by induction, if $y$ is mapped to via the exponent $a$, where $a > 2$, then it is mapped to by every exponent less than $a$ of the same parity. □

Proof of part (b):
We now show that for each of the two parities there exists a range element that is mapped to by every exponent of that parity.

1. Fix a range element $y$, and suppose that $x$ maps to $y$ via the exponent $a$. Now $a$ is either even or odd, hence $a = 2n + h$, where $h$ is either 0 or 1. Since $y = (3x + 1)/2^a$, it follows that $(2^ay) = 3x + 1$. Reduce the equation mod 3, and we get $(2^h)y \equiv 1 \mod 3$, by the following reasoning:

$(2^a)y \equiv 1 \mod 3$ implies $(2^{2n+2h})y \equiv 1 \mod 3$ implies $2^{2n} 2^hy \equiv 1 \mod 3$ because $2^{2n} = 4^n \equiv 1 \mod 3$.

2. Since $y$ is fixed, either $y \equiv 1$ or $y \equiv 2 \mod 3$. (We know that $y$, a range element, is not a multiple of 3 by “Lemma 4.0: Statement and Proof” on page 28). If $y \equiv 1 \mod 3$, then we have $2^h(1) \equiv 1 \mod 3$, which implies that $h$ must be 0. If $y \equiv 2 \mod 3$, then we have $(2^h)(2) \equiv 1 \mod 3$, implying that $h$ must be 1. □

**Lemma 7.0: Statement and Proof**

Let $y$ be a range element of the $3x + 1$ function. Then for each finite exponent sequence $A$, there exists an $x$ that maps to $y$ via $A$ possibly followed by a “buffer” exponent. (For example, if $y$ is mapped to by even exponents, and our exponent sequence $A$ ends with an odd exponent, then there must be a “buffer” exponent following $A$, and similarly if $y$ is mapped to by odd exponents and $A$ ends with an even exponent. However, there are other cases in which a “buffer” exponent is required.)

**Proof:**
1. Each range element $y$ is mapped to by all exponents of one parity (“Lemma 6.0: Statement and Proof” on page 30).

2. Each range element $y$ is mapped to by a multiple of 3 (“Lemma 5.0: Statement and Proof” on page 28).

Each range element is mapped to by an infinity of range elements (“Lemma 5.0: Statement and Proof” on page 28).
3. Let \( y \) be a range element and let \( S = \{ s_1, s_2, s_3, \ldots \} \) be the set of all odd, positive integers that map to \( y \) in one iteration of the \( 3x + 1 \) function. In other words, \( S \) is the set of all elements in a “spiral”. Furthermore, let the \( s_i \) be in increasing order of magnitude. It is easily shown that \( s_{i+1} = 4s_i + 1 \).

(In Fig. 18, \( y = 13 \), \( S = \{ 17, 69, 277, 1109, \ldots \} \))

4. If \( s_i \) is a multiple of 3, then \( 4s_i + 1 \) is mapped to, in one iteration of the \( 3x + 1 \) function, by all exponents of even parity.

To prove this, we need only show that \( x \) is an integer in the equation

\[
4(3u) + 1 = \frac{3x + 1}{2^2}
\]

Multiplying through by \( 2^2 \) and collecting terms we get

\[
(48u) + 4 = 3x + 1
\]

and clearly \( x \) is an integer.
5. If $s_j$ is mapped to by all even exponents, then $4s_j + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of odd parity. 
(The proof is by an algebraic argument similar to that in step 4.)

6. If $s_k$ is mapped to by all odd exponents, then $4s_k + 1$ is a multiple of 3. 
(The proof is by an algebraic argument similar to that in step 4.)

7. The Lemma follows by an inductive argument that we now describe.

Let $y$ be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

$$...3, 2, 1, 3, 2, 1, ...$$

where

“3” means “this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer”; 
“2” means “this odd, positive integer is mapped to by all even exponents”; 
“1” means “this odd, positive integer is mapped to by all odd exponents”.

Each type “2” and type “1” odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

$$...3, 2, 1, 3, 2, 1, ...$$

where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element $y$, and for each finite sequence of exponents $B$, we can find a finite path down through the infinitary tree we have just established, starting at the root $y$. The path will end in an odd, positive integer $x$. Let $A$ denote the path $B$ taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus $y$ is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument. □

$\Box$
Appendix B — A Possible Strategy for A Second Proof of the $3x + 1$ Conjecture

In the above proof of the Conjecture, we argued in what we might call the “vertical” direction, that is, the direction of increasing $i$. We showed that the set of all $35$-level elements of all $35$-level tuples in all $35$-level tuple-sets is the same regardless if counterexamples exist or not, and from that, via an elementary inductive argument, we showed that the contents of all tuple-sets are the same regardless if counterexamples exist or not, which implies that counterexamples do not exist.

In the following possible strategy, we argue in what we might call the “horizontal” direction.

1. Assume counterexamples exist, and consider the set of all first $35$-level tuples in the set of all $35$-level tuple-sets. By what we established in the above proof of the Conjecture, we know that all these tuples are non-counterexample tuples. By Lemma 8.8 in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com, we know that all these tuples are the same regardless if counterexamples exist or not.

2. Now, moving in the “horizontal” direction in any of these tuple-sets, that is, in the direction of increasing first elements of tuples, starting from the first $35$-level tuple, we see from Table 1, “Distances between elements of tuples consecutive at level $i$,” on page 10, that each successive tuple remains the same regardless if counterexamples exist or not, because the Table is not a function of the truth or falsity of the $3x + 1$ Conjecture.

So we must conclude that all the tuples in the tuple-set remain the same regardless if counterexamples exist or not. But this implies that the contents of all tuple-sets remain the same, regardless if counterexamples exist or not, and therefore we must conclude, as we did in our proof of the Conjecture, that counterexamples do not exist.
Appendix C — Possible Strategies for a Proof of the $3x + 1$ Conjecture Based on the 1-Tree

Preliminary Remarks

We must confess that, after several months of sustained effort, we have been unable to find a proof of the $3x + 1$ Conjecture via the 1-tree, which is the tree containing all and only the odd, positive integers that map to 1. (For definitions and background on the 1-tree, see “The 1-Tree” on page 12.) The main reason seems to be the fact that the 1-tree shows only part of the behavior of the $3x + 1$ function, whereas tuple-sets show all of the behavior.

But we may have been overlooking facts and ideas of fundamental importance, hence this section of our paper.

Possible Strategy for 1-Tree-Based Proof: Induction

1. By computer test, we know that all odd, positive integers less than $10^{18}$ are non-counterexamples.

2. We may therefore hope to find an inductive argument by letting $x$ denote the largest odd, positive integer that has been found, by computer test, to be a non-counterexample, and then attempting to show that $x + 2$ is likewise a non-counterexample.

However, there does not seem to be a systematic way to find $x + 2$. Thus, for example, 5 maps to 1. However, to find 7, we must find, by trial and error, that it is an element of the tuple $<7, 11, 13, 5, 1>$. It is true that 9 is an element of the tuple $<9, 7, 11, 13, 5, 1>$, as are 11 and 13. But 15 is an element of the tuple $<15, 23, 35, 53, 5>$, which we must find by trial and error.

But we must not fail to point out that all, odd, positive integers are present in the “spiral” $S = \{1, 5, 21, 85, 341, \ldots \}$ and in the intervals between the elements! (Recall that $S$ is the set of odd, positive integers that map to 1 in a single iteration of the $3x + 1$ function.)

Possible Strategy for 1-Tree-Based Proof: Show That Counterexamples Eliminate Exponent Sequences That Cannot Be Eliminated

1. We must remember that there is exactly one 1-tree, whether or not counterexamples exist (see “Lemma 3.0: Statement and Proof” on page 27). If the set of all “spiral” elements in the 1-tree is the set of all odd, positive integers, then counterexamples do not exist. Otherwise, counterexamples exist.

2. If a counterexample exists, it is the first element of a 2-tuple in a 2-level tuple-set. However, by what we have said regarding the table in the section “Properties of the 1-Tree” on page 14, this means that the counterexample is an element of a counterexample “spiral”. The “spiral” maps to a range-element $y$ of the $3x + 1$ function in a single iteration of the function, and therefore $y$ is the root of a $y$-tree. The $y$-tree contains an infinity of counterexample “spiral”s.

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)}$, have been tested and found to be non-counterexamples.
If a counterexample exists, there is an infinity of counterexamples ("Lemma 2.0" on page 11), and the fact that, by the ... 2, 1, 3, 2, 1, 3, ... pattern in “spiral” elements (see “Properties of the 1-Tree” on page 14)) we know that there is an infinity of counterexample range-elements. Hence there is an infinity of counterexample \( y \)-trees, where \( y \) is a counterexample range element).

3. There are no counterexample tuples in the set of all tuple-sets if the set contains only non-counterexample tuples.

It is easily shown that in order for there to be an infinity of tuples associated with each finite exponent sequence there must be an infinity of non-counterexample tuples in each tuple-set that are extended by the exponent 1, and an infinity that are extended by the exponent 2, etc. (See “Extensions of Tuple-sets” on page 9.) However, this property cannot survive the omission of a countable infinity of “spiral”s.

It seems, therefore, that we must conclude that the existence of a counterexample results in the failure of non-counterexample tuples to be associated with each finite exponent sequence, contradicting “Lemma 2.0” on page 11.

If this reasoning can be made precise and valid, we have a proof of the 3x + 1 Conjecture.

(Note: we must reconcile the omission of non-counterexample “spiral”s from tuple-sets, with the fact that the 1-tree is the same, whether or not counterexamples exist.)

Possible Strategy for a 1-Tree-Based Proof: Color All Elements of Tree Derived From Computer Tests (First Attempt)

1. Let \( S \) denote the “spiral” \( \{1, 5, 21, 85, 341, \ldots \} \) in the 1-tree. (For background on “spiral”s see “Recursive ‘Spiral’ ‘S: The Structure of the 3x + 1 Function in the ‘Backward’, or Inverse, Direction” on page 12.)

   This “spiral” is the set of odd, positive integers that map to 1 in one iteration of the 3x + 1 function. The set of odd, positive integers lying between “spiral” elements is called an interval. Thus, \( \{3\} \) is the first interval in \( S \), \( \{7, 9, 11, 13, 15, 17, 19\} \) is the second interval, etc.

2. By computer test\(^1\), we know that all odd, positive integers less than \( 5 \cdot 10^{18} \), are non-counterexamples. These integers occupy more than the first 30 successive intervals in \( S \).

   Call the largest odd, positive integer less than \( 5 \cdot 10^{18} \), \( w \).

3. A basic fact governing “spiral”s is that the elements of each “spiral” other than \( S \), occupy an infinity of successive intervals in \( S \). Thus, consider the “spiral” \( \{3, 13, 53, 213, \ldots \} \) and the intervals in the “spiral” \( S \).

   We remark in passing that each “spiral” element \( x \), and the range element \( y \) it maps to in one iteration of the 3x + 1 function, is a tuple \( <x, y> \) in a tuple-set. Since the set of elements of a “spiral” map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all “spiral”s in the 1-tree is the set of all first elements of all non-

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\(^1\) See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than \( 20 \cdot 2^{58} \approx 5.76 \cdot 10^{18} \), which is greater than \( 3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)} \), have been tested and found to be non-counterexamples.
counterexample 2-tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

4. Therefore we can go through 1, 2, 3, ..., w. Call the set of these odd, positive integers, W. We can color green each interval element y in W that is a range element of the 3x + 1 function, and then color green each element of the y-tree (same structure as the 1-tree, except with y as its root — see the above-mentioned section on “spiral”s). We can then color green the elements of all extensions of each y in W, stopping at 1.

We can then color green each interval element v in W that is not a range-element (that is, which is a multiple-of-3), and all extensions of each such v, stopping at 1.

We can then color green each interval element in S that equals an element that we have already colored green. Step 3 assures us that infinities of interval elements in S greater than w will be colored green.

5. Counterexamples occupy intervals only. They are never “spiral” elements.

If counterexamples exist, they can never themselves fill an interval in S as non-counterexamples less than w have done. The reason is that this would prevent non-counterexample “spiral” elements from having “room” in the interval, as these elements must, by step 3.

But after the first counterexample, non-counterexamples can never themselves fill an interval in S as non-counterexamples have done. The reason is that this would prevent counterexample elements from having room in the interval, as these elements must, by step 3.

So after the first counterexample, all subsequent intervals in S must be shared between non-counterexamples and counterexamples. There can never again be an interval containing solely non-counterexamples, and there can never be one containing solely counterexamples.

It seems hard to believe that the behavior of the 3x + 1 function would change so radically once the first counterexample has appeared — especially since a counterexample is simply another odd, positive integer that, if it and all other counterexamples were non-counterexamples, would result in an infinite succession of intervals containing solely non-counterexamples.

We might ask, informally, “How does the function ‘know’ what to do when it is given an odd, positive integer that may or may not be a counterexample?” Keeping in mind that, as far as the function is concerned, a counterexample is simply an odd, positive integer that is not in the 1-tree, one answer (invalid!) is the following.

For each odd, positive integer the function is given, it asks:

1. Is this integer in the 1-tree?
   If the answer is yes, then the function asks,
   Have I been given an integer not in the 1-tree before?
   If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule;
   If the answer is no, then the function continues its practice of filling successive intervals entirely with non-counterexamples.

2. If the answer to 1. is no, then the function asks,
   Have I been given an integer not in the 1-tree before?
If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule;
If the answer is no, then the function begins the permanent application of its mixed-intervals-only rule.

But of course there is nothing like this question/answer sequence in the definition of the $3x + 1$ function.

Whether all this can lead to a proof of the $3x + 1$ Conjecture, remains to be seen.

**Possible Strategy for a 1-Tree-Based Proof: Color All Elements of Tree Derived From Computer Tests (Second Attempt)**

1. If a counterexample exists, then an infinity of counterexamples exists (“Lemma 2.0: Statement and Proof” on page 27). So if a counterexample exists, an infinity of odd, positive integers is not in the 1-tree.

2. By computer test\(^1\), we know that all odd, positive integers less than $10^{18}$ are non-counterexamples. Let $w$ denote the largest of these odd, positive integers.

3. We can color green all “spiral” elements that must be in the 1-tree as a result of all consecutive odd, positive integers in the range 1 through $w$ being non-counterexamples.

   *If counterexamples do not exist*, then all these “spiral” elements are part of the 1-tree.

4. Let $S$ denote the “spiral” \(\{1, 5, 21, 85, 341, \ldots\}\) in the 1-tree. This is the set of all odd, positive integers that map to 1 in one iteration of the $3x + 1$ function.

   *If counterexamples do not exist*, then all the “spiral” elements of each $y$-tree, where $y$ is a range element in $S$ that is greater than $10^{18}$, are part of the 1-tree.

4. We conclude that counterexamples do not exist, by the following reasoning:

   Let $p$ denote “counterexamples do not exist”.
   Let $q$ denote our conclusion in step 3.
   Let $q'$ denote our conclusion in step 4.

   Then the statement

---

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)}$, have been tested and found to be non-counterexamples.
(1)

\((p \implies q) \text{ and } (p \implies q') \text{ and } q \text{ and } q') \implies p\)

if \(q \text{ and } q'\) are both true, can only be valid if \(p\) is true.

The argument is faulty because in fact \(((p \text{ or } \lnot p) \implies q')\). However, if \(\lnot p\) is true (that is, \(p\) is false) the statement (1) is also valid.
Appendix D — For Professional Mathematicians Only

Understandable Reluctance of Mathematicians to Read This Paper

There has been an understandable reluctance on the part of professional mathematicians to give serious attention to this paper, or its predecessors. It seems clear to us that the main reason is mathematicians’ difficulty in believing that such an extraordinarily difficult problem can have been solved by a non-mathematician (our degree is in computer science, and we have spent most of our working life doing research in the computer industry). This skepticism is reinforced by the fact that there have been many false claims of solutions to the $3x + 1$ Problem, the overwhelming majority of which having been made by non-mathematicians.

But we must point out that the occasionally-heard remark, “Nothing of importance in mathematics has ever come from outside the university”, is, in fact, false, considering that some of the best of the best worked entirely outside the university — Descartes, Pascal, Fermat, Leibniz, and Galois, to name only the most famous.

We must also not fail to mention another reason for mathematicians’ reluctance to read this paper, and that is the online presence of obsolete criticisms of the paper. For example, Stack Overflow has a website containing criticisms of a proof in a 2015 version of the paper. Not only were the criticisms false, but the proof that was criticized has long since been removed from the paper. This website appears next to the website containing this paper, and thus unquestionably discourages potential readers — especially mathematicians — from reading this paper. Yet despite many pleas on our part, the managers of the website have refused to delete the criticisms or to add a note to the website stating that the criticisms do not apply to the current version of the paper. Nor have they explained to us the reason for their refusals. Apparently we have no recourse in this matter, except to encourage others to boycott the Stack Overflow websites, and to write to the organization explaining the reason for the boycott. The email address is team@stackoverflow.com, the item no. is 201708202111462820.

It appears that the managers of the above website have no experience of actually doing research. They believe that if a paper is published online and contains an error, that means that the author is incapable of correcting the error, and that his underlying ideas do not deserve any attention. But errors are almost inevitable in the course of attempting to solve very difficult problems. We remind the reader that Wiles’ first proposed proof of the Taniyama–Shimura–Weil Conjecture in the early 90s, which implied a proof of Fermat’s Last Theorem, contained an error that took Wiles, with the help of the mathematician Richard Taylor, more than a year to repair. The important question obviously was, Do the underlying ideas in this paper offer hope for correcting the error? And the answer was yes.

We have been struck by the eagerness with which readers of this paper look for anything they can regard as an error, and the indifference they display to understanding, and thinking about, the underlying ideas.

If You Do Not Accept Our Proofs of the $3x + 1$ Conjecture...

If you do not accept our proofs of the Conjecture, or any of the possible strategies for a proof that are set forth in the above appendices, we urge you to at least peruse our paper, “Are We Near a Solution to the $3x + 1$ Conjecture?” on occampress.com. This paper contains a wealth of results, insights, possible strategies for a proof, plus a section on what we have called “3x + 1-like functions”. We will welcome comments.
We are confident that at least two publishable, significant papers can be produced from the material in our $3x + 1$ papers, and that this is true even if the proof of the Conjecture in the present paper and all the possible strategies in the appendices, are faulty and cannot be repaired.

We feel that the two structures we have discovered that underlie the $3x + 1$ function, namely, tuple-sets and recursive “spiral”s, are of fundamental importance, and should be brought to the attention of the entire $3x + 1$ research community.

**Difficulty, So Far, In Getting This Paper Published**

Not surprisingly, so far, no journal that we know of is willing to even consider this paper for possible publication. The reason seems to be that editors cannot believe that such a difficult problem might have been solved by a non-mathematician.

**Incentives for Mathematicians to Take This Paper Seriously and to Spread the Word About It**

Unquestionably, if this paper contains a solution to the $3x + 1$ Problem, or can easily be modified to contain a solution, considerable prestige will be gained by the first mathematicians who promote the paper. Of course, mathematicians who believe that the proof of the Conjecture is correct, and/or that at least one of the possible strategies in the appendices look promising, but do not want to risk their reputations by saying so, especially given that the author of the paper is not an academic mathematician, will not be recipients of that prestige. We are offering three incentives:

1. Any reasonable consulting fee;
2. Generous mention in the Acknowledgments when the paper is published. (But no name will be mentioned without the prior written approval of the mathematician concerned.)
3. An offer of shared authorship to the first mathematician who makes a significant contribution to the paper prior to publication.

In any case, all communications we receive about this paper will be kept strictly confidential.
Appendix E — A Brief Summary of Our First Proof of the 3x + 1 Conjecture

The 3x + 1 Problem asks if repeated iterations of the function \( C(x) = (3x + 1)/(2^a) \) always terminate in 1. Here \( x \) is an odd, positive integer, and \( a = \text{ord}_2(3x + 1) \), the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the 3x + 1 Conjecture.

(Note: the reader is asked to inform us of the first sentence that the reader believes contains an error, and what that error is.)

The following is a summary of our solution to the Problem — that is, of our first proof of the 3x + 1 Conjecture.

1. Using our definition\(^1\) of the 3x + 1 function, we define an infinite set of infinite \( i \)-level tuple-sets, where \( i \geq 2 \), and where a tuple is the result of a finite sequence of iterations of the function. Thus, for example, \(<17, 13, 5, 1>\) is a tuple in a tuple-set.

2. By computer test, we know that the 35-level elements in all first 35-level tuples in all 35-level tuple-sets are non-counterexamples. This fact holds regardless if counterexamples exist or not -- that is, no known non-counterexample can somehow become a counterexample, under any circumstances.

3. The distance function (part (a) of “Lemma 1.0: the “Distance” Functions \( d(i, i) \) and \( d(1, i) \)” on page 9) gives the value of the \( i \)th element of the \( n \)th \( i \)-level tuple in an \( i \)-level tuple-set, where \( n \geq 2 \). This function is not a function of the truth or falsity of the 3x + 1 Conjecture.

4. Therefore, from step 2, the value of the 35th-level element of the \( n \)th 35-level tuple in any 35-level tuple-set is the same, whether or not counterexamples exist.

5. But since each \((i + 1)\) - level tuple-set is an extension of \( i \)-level tuples in an \( i \)-level tuple-set, we see that the value of the 36th-level element in the \( n \)th 36-level tuple in any 36-level tuple-set is the same, whether or not counterexamples exist. And so on for all \( i \)-level tuple-sets, where \( i \geq 36 \).

6. But then we must arrive at the conclusion that the set of all non-counterexample tuples if counterexamples do not exist, is the same as the set of all non-counterexample tuples if counterexamples exist. Therefore the set of counterexamples is the empty set, and the 3x + 1 Conjecture is true.

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\(^1\) It is actually Crandall's alternate, but equivalent, definition of the function
Appendix H — Second Proof of the $3x + 1$ Conjecture

The following proof of the $3x + 1$ Conjecture is based on the idea underlying the proof of ("Theorem: The $3x + 1$ Conjecture is true." on page 19). Like that one, it shows that the set of all tuple-sets (structure and contents) is the same, whether or not counterexamples exist. This implies that, if counterexamples exist, there is a contradiction, hence counterexamples do not exist.

(Note: we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

1. **Definitions:** an anchor tuple is the first $i$-level tuple in an $i$-level tuple-set, where $i \geq 2$. An anchor is the $i$-level element of the anchor tuple in an $i$-level tuple-set.

(a) If $x$ is a range element of the $3x + 1$ function, then $x$ is eventually — for some $i \geq 2$ — an anchor

Proof: If $x$ exists, then for some $i \geq 2$, $x < 2 \cdot 3(i-1)$. Therefore, $x$ is an anchor (part (a) of “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 9). □

It follows trivially that $x$ is also an anchor for all greater $i$. ("Once an anchor, always an anchor.")

(b) If $x$ is a non-counterexample anchor, then it is a non-counterexample anchor whether or not counterexamples exist.

Proof: The arithmetic defining the $3x + 1$ function is not itself a function of the truth or falsity of the $3x + 1$ Conjecture. □

Thus, for example, 13 is a non-counterexample (maps to 1) today, and if the Conjecture is proved true tomorrow, it will be a non-counterexample tomorrow, and if the Conjecture is proved false tomorrow it will still be a non-counterexample.

(Actually, the statement (b) holds for non-counterexamples in general, not just non-counterexample anchors.)

2. At this point, it is reasonable to assume that there are two possible sets of anchors: one containing counterexamples if counterexamples exist, and one not containing counterexamples, if counterexamples do not exist.

However this assumption is false.

(1) There is one and only one set of anchors, regardless if counterexamples exist or not.

Proof: 

(a) The “distance” between consecutive $i$-level elements of an $i$-level tuple-set is $2 \cdot 3^{(i-1)}$ (follows from part (a) in “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 9).

Thus, for example, the distance between the first and second 2-level elements of any 2-level tuple-set having 1 as first element, namely, between the elements 1 and 7, is $2 \cdot 3^{(2-1)} = 2 \cdot 3^1 = 6$. The distance between the second and third elements, that is, between the elements 7 and 13, is
likewise 6.

(b) Each \( i \)-level anchor is less than \( 2 \cdot 3^{(i-1)} \) (follows from part (a) in “Lemma 1.0: the “Distance” Functions \( d(i, i) \) and \( d(1, i) \)” on page 9). Of course, each anchor is greater than 0, by definition of the domain of the \( 3x + 1 \) function.

(c) For each \( i \geq 2 \), the number of anchors in all \( i \)-level tuple-sets is \( 2 \cdot 3^{((i-1)-1)} \) (follows from part (a) in “Lemma 1.0: the “Distance” Functions \( d(i, i) \) and \( d(1, i) \)” on page 9).

Thus, for example, the number of anchors in all 2-level tuple-sets is \( 2 \cdot 3^{((2-1)-1)} = 2 \cdot 3^0 = 2 \). These anchors are 1 and 5. It is easy to show that 1 is mapped to by all even exponents, and 5 is mapped to by all odd exponents. Those are the only two possibilities for the anchors of 2-level tuple-sets.

The number of anchors in all 3-level tuple-sets is \( 2 \cdot 3^{((3-1)-1)} = 2 \cdot 3^1 = 6 \). These anchors are 1, 5, 7, 11, 13, 17.

(d) The set of \((i+1)\)-level anchors comes into being as follows:

If \( a \) is an \( i \)-level anchor then \( a \) is an \((i+1)\)-level anchor, because if \( a \) is less than \( 2 \cdot 3^{(i-1)} \), as it must be if \( a \) is an \( i \)-level anchor, then \( a \) is certainly less than \( 2 \cdot 3^{((i+1)-1)} \).

Since the \( i \)-level tuple-set element \( a + 1 \cdot (2 \cdot 3^{(i-1)}) \) is less than \( 2 \cdot 3^{((i+1)-1)} \), the element is an \((i+1)\)-level anchor.

Since the \( i \)-level tuple-set element \( a + 2 \cdot (2 \cdot 3^{(i-1)}) \) is less than \( 2 \cdot 3^{((i+1)-1)} \), the element is an \((i+1)\)-level anchor.

No other element of an \( i \)-level tuple-set is less than \( 2 \cdot 3^{((i+1)-1)} \), and therefore no other element of an \( i \)-level tuple-set is an \((i+1)\)-level anchor.

The reader can see an example of this increase in anchors from level 2 to level 3 in step 2 (c).

(e) The process we have described is unique. It yields all \( i \)-level anchors for all \( i \geq 2 \). There is thus one and only one set of anchors. In other words:

If counterexamples do not exist, then the set of all anchors is exactly the set that results from the process we have described. Call that set \( S \).

If counterexamples exist, then the set of all anchors is exactly the set that results from the process we have described. In other words, if counterexamples exist, then the set of all anchors is the same set \( S \). \( \Box \)

3. Computer tests\(^1\) have shown the Conjecture to be valid for all consecutive odd positive integers up to at least \( 10^{18} + 1 \), which includes all the anchors (each of which is a non-counterexample anchor) from level 2 through level 35.

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\(^1\) See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than \( 20 \cdot 2^{58} \approx 5.76 \cdot 10^{18} \), which is greater than \( 3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)} \), have been tested and found to be non-counterexamples. These include the set of all 35-level anchors.
A Solution to the $3x + 1$ Problem

(This is not true in the case of the $3x - 1$ function since the first counterexample in the case of that function is 5.)

Furthermore, there is an infinity of non-counterexample anchors at levels greater than 35. Proof: each range element, hence each non-counterexample anchor at levels 2 through 35 is mapped to by an infinity of odd, positive integers (see Lemma 13.0 in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com), and range element there is an infinity of them in that infinity of odd, positive integers is also mapped to by an infinity of odd, positive integers, etc.. The fact that there is an infinity of these range elements in each case means that an infinity of them are greater than the largest anchor at level 35. □

The unique process for generating anchors (step 2) then continues to generate anchors for all levels beyond 35. The set of anchors so generated for each level is the same whether or not counterexamples exist (the process is unique). So, in particular, we can regard the process as generating the set of all non-counterexample anchors.

If counterexamples exist, the set of anchors so generated is the same as the set of anchors if counterexamples do not exist. Each anchor is an element of an infinite tuple. Non-counterexample infinite tuples are, by definition, of the form $<x, \ldots, 1, 1, 1, \ldots>$, whereas counterexample infinite tuples are of the form $<y, \ldots>$, with no element equal to 1.

And so if counterexamples exist, then some counterexample anchors are the same as non-counterexample anchors, which is absurd. Therefore the $3x + 1$ Conjecture is true.

Another way of stating our argument here is:

The set of all tuple-sets (structure and contents) is the same, whether or not counterexamples exist. Therefore there is no difference between the set of all counterexamples and the set of all non-counterexamples. Therefore, counterexample tuples behave exactly the same as non-counterexample tuples, which is absurd. Therefore counterexamples do not exist, and the Conjecture is true. □

Remark

Suppose that the anchors were all and only those odd, positive integers that map to 1. Suppose, further, that if counterexamples exist, they never become anchors. Then there would be no difficulty: the set of anchors would be fixed, whether or not counterexamples existed, and they would all map to 1.

However, the simple argument in step 1 (a) shows that if counterexamples exist, they must eventually be anchors. And so there is, in reality, a difficulty: how to reconcile this fact with the fact that the set of anchors is fixed whether or not counterexamples exist. Our proof, above, shows one way to reconcile this fact.
Appendix I — The 3x – 1- and Related Functions

There is no question but that what we have called the “3x – 1 Test” has helped us to recognize errors in our proposed proofs of the Conjecture. (In this Test, we ask if our proof also proves the 3x – 1 Conjecture. If it does, then our proof is in error, because there are known counterexamples to the 3x – 1 Conjecture, the smallest of which is 5.)

However, there are critics who make the following argument:

(a) There are other “3x + 1-like” functions. (See for example, our paper, “Are We Near a Solution to the 3x + 1 Problem?”, on occampress.com, “Appendix C – ‘3x + 1-like’ Functions”.) There are an infinite number of these. Each is easily shown to have a counterexample (it is an infinite cycle) to the related conjecture for the function.

(b) Your proposed proof applies to each of these conjectures. (What is the justification for this statement?)

(c) Therefore your proof is wrong.

A proof must stand or fall on its own terms. Mathematical logic, and, in particular, computerized proof-checking, would face an insurmountable obstacle if each proof of a conjecture required that the author, or the proof-checker, find all and only the related conjectures (whatever “related” means) that are known to be false, and then prove that the proof in question did not also prove one of those conjectures.

Furthermore, even if our proof of the Conjecture proves none of the related conjectures, each of which is known to be false, that in no way confirms the validity of our proof! There may be other infinite cycles in one or more of the 3x + 1-like functions, and/or there may be computations that never yield 1, although they are not infinite cycles.

Of course, at present, our set of 3x + 1-like functions is not recognized as the only set of functions that have the property of being “like” the 3x + 1 function. At present, as far as we know, at the very least one is free to call any Dx + C function, where D, C are integers, “3x + 1-like”.

A definite contribution to 3x + 1 research would be to show how all these Dx + C functions are related.

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1. Each of the 3x + 1-like (that is, 3x + C) functions, where, here, C is positive), defined in the above-referenced section, has an infinite cycle at C. Thus, for example, 3 starts an infinite cycle in the 3x + 3 function, 5 starts an infinite cycle in the 3x + 5 function, 13 starts an infinite cycle in the 3x + 13 function, etc.

However, in the case of the 3x + 1 function, the infinite cycle is the same as the trivial cycle, namely, {1, 1, 1, ...}. In other words, these infinite cycles “converge” to 1 in the case of the 3x + 1 function, as C decreases.